Efficient Risk Estimation for the Credit Valuation Adjustment

Abdul-Lateef Haji-Ali
Joint work with Mike Giles and Jonathan Spence

Heriot-Watt University

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Motivation: CVA Capital Charge

$$\text{Adjusted Value}_t = \text{Risk-Neutral Value}_t - \mathrm{CVA}_t$$

- The value of a financial portfolio at time t is reduced according to the credit valuation adjustment CVA_t, to account for the possibility of counterparty default.
- Through the credit valuation adjustment, a financial institution can observe losses due to the event that a counterparty becomes more likely to default - responsible for two thirds of losses from counterparty risk factors in 2008 financial crisis.
- Capital charge based on a value-at-risk formula introduced in Basel III accords to mitigate such risks.

Motivation: CVA Capital Charge (Nested Simulation)

- Given market at time 0 - simulate $\mathcal{G}_H$-measurable market and credit risk factors under the physical measure P at short risk horizon $0 < H \ll 1$.
  - Given risk factors at time H, simulate instances of default τ under the risk-neutral measure Q occurring before contract maturity $T > 0$.
    - Given the (risk-neutral) market state at default time $\tau < T$, simulate random losses $\pi(S_T)$ based on the asset values S_T under the risk-neutral measure.

Value-at-Risk formula:

$$\varphi = \mathsf{P}\left[\frac{\mathrm{CVA}_H}{B_H} - \mathrm{CVA}_0 > \lambda_\varphi\right],$$

$$\mathrm{CVA}_t = B_t\,\mathsf{E}^{\mathsf{Q}}\left[\chi_{t\leq\tau\leq T}\,\mathrm{LGD}\max\left\{\mathsf{E}^{\mathsf{Q}}\left[\,B_T^{-1}\pi(S_T)\,\middle|\,\mathcal{G}_\tau\,\right],0\right\}\,\middle|\,\mathcal{G}_t\right].$$

Overview

Let X, Y, Z be random variables and $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be Lipschitz in both arguments. Consider the system

$$\begin{aligned} \varphi &= \mathrm{P}[\, U_0(Z) > \lambda_\varphi \,] \\ U_0(Z) &= \mathrm{E}[\, f(U_1(Y), Y) \,|\, Z \,] \\ U_1(Y) &= \mathrm{E}[\, X \,|\, Y \,]. \end{aligned}$$

Key features:

- Recursive approximation of nested expectations $U_0(Z)$ and $U_1(Y)$, paired with approximation of the variables X, Y and Z.
- Approximation of discontinuous observables:

$$\varphi = \mathrm{P}[\, Q > 0 \,] = \mathrm{E}[\, \chi_{Q>0} \,].$$

Nested Monte Carlo simulation has $\mathcal{O}(\varepsilon^{-5})$ cost to attain an accuracy ε.

Multilevel Monte Carlo

Want to approximate

$$\mathsf{E}[\,Q\,]$$

given approximate samples $Q \approx Q_\ell$, with

$$\begin{aligned}\mathrm{Cost}(Q_\ell) &\propto 2^{\gamma\ell}\\ |\mathsf{E}[\,Q - Q_\ell\,]| &\propto 2^{-\alpha\ell}\\ \mathrm{Var}[\,Q - Q_\ell\,] &\propto 2^{-\beta\ell}.\end{aligned}$$

Then, let

$$\begin{aligned}\mathsf{E}[\,\Delta_\ell Q\,] &= \begin{cases}\mathsf{E}[\,Q_\ell - Q_{\ell-1}\,] & \ell > 0\\ \mathsf{E}[\,Q_0\,] & \ell = 0\end{cases}\\ \mathsf{E}[\,Q\,] \approx \mathsf{E}[\,Q_L\,] &= \sum_{\ell=0}^{L} \mathsf{E}[\,\Delta_\ell Q\,].\end{aligned}$$

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The cost of attaining root mean square error ε is of order

$$\varepsilon^{-2} \begin{cases} 1 & \beta > \gamma \\ |\log \varepsilon|^2 & \beta = \gamma \\ \varepsilon^{-(\gamma-\beta)/\alpha} & \beta < \gamma. \end{cases}$$

Unbiased Multilevel Monte Carlo [Rhee, Glynn, 2015]

$$\mathsf{E}[\,Q\,] = \sum_{\ell=0}^{\infty} \mathsf{E}[\,\Delta_\ell Q\,] = \mathsf{E}\Big[\,(\Delta_\kappa Q)\,2^{\zeta\kappa}/\,C_\zeta\,\Big]$$

where κ is a random, non-negative integer with probability mass

$$\mathsf{P}[\,\kappa = \ell\,] = C_\zeta 2^{-\zeta\ell}.$$

Provided,

$$\mathrm{Cost}(Q_\ell) \propto 2^{\gamma\ell}$$

$$\mathsf{E}[\,|Q - Q_\ell|^q\,] \propto 2^{-q\beta\ell/2},$$

$(\Delta_\kappa Q)\,2^{\zeta\kappa}/\,C_\zeta$ has finite expected sampling cost and p^{th} moment when

$$\gamma < \zeta < \frac{p}{p-1}\frac{\beta}{2} \implies p < \min\left\{q, \frac{1}{1-\beta/2\zeta}\right\}.$$

Example 1: $\beta = 2\gamma = 2, \zeta = (\beta+\gamma)/2 = 3/2, q \to \infty \Rightarrow p < 3.$
Example 2: Same, with
$q = 3 - \varepsilon, \zeta < (3-\varepsilon)/(2-\varepsilon) \Rightarrow p < 3 - \varepsilon.$

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$$\begin{aligned} \varphi &= \mathrm{P}[\, U_0(Z) > \lambda_\varphi \,] \\ U_0(Z) &= \mathrm{E}[\, f(U_1(Y), Y) \mid Z \,] \\ U_1(Y) &= \mathrm{E}[\, X \mid Y \,]. \end{aligned}$$

Key features:

- Recursive approximation of nested expectations $U_0(Z)$ and $U_1(Y)$, paired with approximation of the variables X, Y and Z.
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$$\varphi = \mathrm{P}[\, Q > 0 \,] = \mathrm{E}[\, \chi_{Q>0} \,].$$

Nested Simulation

Consider now the nested pair of expectations

$$U_0 := \mathrm{E}[\, f(U_1(Y),\; Y)\,]$$
$$U_1(Y) := \mathrm{E}[\, X \mid Y \,]$$

Given exact samples of X and Y: Multilevel Monte Carlo with antithetic nested Monte Carlo averages for $\mathrm{E}[\, X \mid Y \,]$.
[Bourgey, De Marco, Gobet, 2020], [Bujok, Hambly, Reisinger, 2015], [Giles, H-A, 2019].

Problem: Exact samples of X and Y are not available. Instead, given $Y = y$ we approximate $X \approx X_k(y)$ using a Milstein scheme. Similarly, $Y \approx Y_\ell$.

Solution: Combine nested 'inner' multilevel Monte Carlo estimate of $U_1(y)$, given $Y = y$, within an 'outer' multilevel Monte Carlo estimate of U_0.

Antithetic Multilevel Difference

Consider now the nested pair of expectations

$$U_0 := \mathrm{E}[\, f(U_1(Y),\ Y)\,]$$
$$U_1(Y) := \mathrm{E}[\, X \mid Y\,]$$
$$\widehat{U}_{1,\ell}(y) := \sum_{k=0}^{\ell} \frac{1}{N_{\ell,k}} \sum_{n=1}^{N_{\ell,k}} \Delta_k^{(n)} X(y)$$
$$N_{\ell,k} \propto 2^{\ell-k}.$$

Antithetic multilevel difference:

$$\Delta_\ell f := f\Big(\widehat{U}_{1,\ell}(Y_\ell), Y_\ell\Big) - \frac{1}{2}\sum_{i=0}^{1} f\Big(\widehat{U}^{(i)}_{1,\ell-1}(Y_{\ell-1}), Y_{\ell-1}\Big),$$

where

$$\widehat{U}_{1,\ell}(y) - \frac{1}{2}\sum_{i=0}^{1} \widehat{U}^{(i)}_{1,\ell-1}(y) = \mathcal{O}(\Delta_\ell X).$$

Convergence

Theorem ([H-A, Spence, 2023])

Assume f is piecewise-twice differentiable and with bounded first and second derivative, and that for $\beta \geq 1$ and $q \geq 2$

$$\begin{aligned} \mathit{Cost}(X_k(\cdot)) + \mathit{Cost}(Y_\ell) &\propto 2^k + 2^\ell \\ \mathrm{E}[\, \|X_k(Y_\ell) - X_{k-1}(Y_\ell)\|^q \,] + \mathrm{E}[\, \|Y_\ell - Y_{\ell-1}\|^q \,] &\propto 2^{-q\beta k/2} + 2^{-q\beta\ell/2} \\ \mathrm{E}[\, \|X_k(Y) - X_k(Y_\ell)\|^q \,] &\propto 2^{-q\beta\ell/2}. \end{aligned}$$

Then,

$$\begin{aligned} \mathrm{Cost}(\Delta_\ell f) &\propto \ell 2^\ell \\ \mathrm{Var}[\, \Delta_\ell f \,] &\propto 2^{-\min\{\beta, 3q/2(q+1)\}\ell}. \end{aligned}$$

Consequently, the cost of estimating U_0 to accuracy ε is of order

$$\varepsilon^{-2} \begin{cases} 1 & \beta > 1 \\ |\log \varepsilon|^3 & \beta = 1. \end{cases}$$

Extensions

- Can be applied recursively to consider repeatedly nested expectations of the form

$$U_j(Y_j) = \mathrm{E}[\, f_{j+1}(U_{j+1}(Y_{j+1}), Y_{j+1}) \mid Y_j \,]$$
$$U_{T-1}(Y_{T-1}) = \mathrm{E}[\, f_T(Y_T) \mid Y_{T-1} \,].$$

  - Bermudan option pricing/optimal control.

- Can randomise the approximation level ℓ in the terms $\Delta_\ell X$ and $\Delta_\ell f$ to obtain unbiased estimates of $U_1(Y)$ and U_0.
  - [Zhou, Wang, Blanchet, Glynn, 2022], [Syed, Wang, 2023].
  - Reduces the number of finite moments - leading to large variance and sampling cost for repeatedly nested expectations as above.
- Can be extended to include antithetic path simulation of Y.

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Key features:

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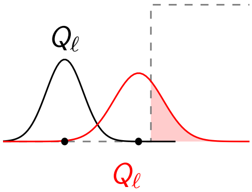
Framework

$$\mathrm{P}[\,Q>0\,]=\mathrm{E}[\,\chi_{Q>0}\,]\approx\mathrm{E}[\,\chi_{Q_0>0}\,]+\sum_{\ell=1}^{L}\mathrm{E}\left[\,\chi_{Q_\ell>0}-\chi_{Q_{\ell-1}>0}\,\right]$$

Theorem

For root mean square error ε and (random), positive-valued, normalising factor σ_ℓ

$$\left.\begin{array}{r}Cost(Q_\ell)\propto 2^{\gamma\ell}\\ \mathrm{E}\left[\,|Q-Q_\ell|^q\sigma_\ell^{-q}\,\right]\propto 2^{-q\beta\ell/2}\\ \mathrm{P}[\,|Q_\ell/\sigma_\ell|\leq x\,]\propto x\end{array}\right. \implies \underset{\text{Cost}}{\text{MLMC}}\propto\begin{cases}\varepsilon^{-2} & \beta>\frac{q+1}{q}2\gamma\\ \varepsilon^{-2}|\log\varepsilon|^2 & \beta=\frac{q+1}{q}2\gamma\\ \varepsilon^{-1-2(\frac{q+1}{q})(\frac{\gamma}{\beta})} & \beta<\frac{q+1}{q}2\gamma.\end{cases}$$



Previous Research

- **Explicit smoothing $\chi_{x>0} \approx g(x)$:**
  [Giles, Nagapetyan, Ritter, 2015].
- **Numerical smoothing:**
  [Bayer, Hammouda, Tempone, 2023], [Giles, Debrabant, Rößler, 2019].
- **Path branching:**
  [Giles, H-A, 2022].
- **Quasi-Monte Carlo:**
  [Xu, He, Wang, 2020].
- **Adaptivity:**
  - For partial differential equations with random coefficients [Elfverson, Hellman, Målqvist, 2016].
  - For nested expectations $Q = \mathrm{E}[\, X \mid Y \,]$ [Broadie, Du, Moallemi, 2011], [Giles, H-A, 2019].

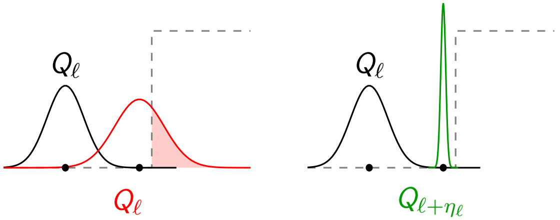
Adaptivity

$$\mathrm{E}[\,\chi_{Q>0}\,] \approx \mathrm{E}[\,\chi_{Q_0>0}\,] + \sum_{\ell=1}^{L} \mathrm{E}\Big[\,\chi_{Q_{\ell+\eta_\ell}>0} - \chi_{Q_{\ell-1+\eta_{\ell-1}}>0}\,\Big]$$

Theorem ([H-A, Spence, Teckentrup, 2022])

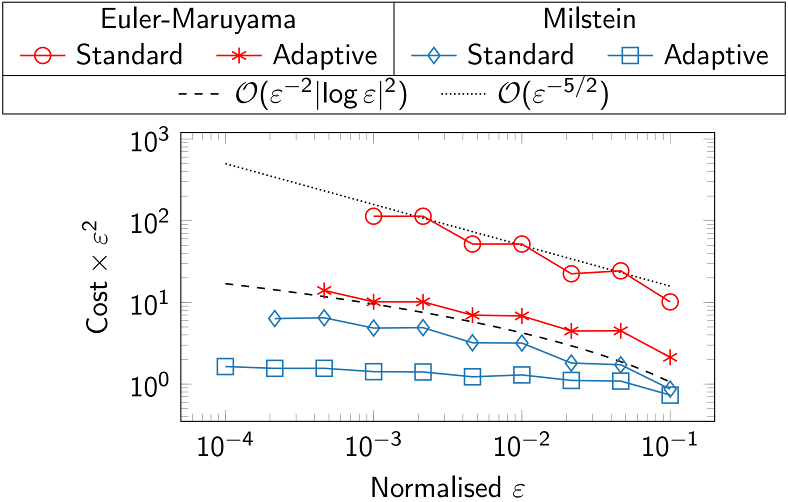
Let η_ℓ be such that $|Q_{\ell+\eta_\ell}| \geq \sigma_{\ell+\eta_\ell} 2^{\gamma(\ell(1-r)-\eta_\ell)/r}$ or $\eta_\ell = \ell$. Then, for a root-mean-square error $\varepsilon > 0$

$$\left.\begin{array}{r} \mathit{Cost}(Q_\ell) \propto 2^{\gamma\ell} \\ \mathrm{E}\Big[\,|Q - Q_\ell|^q \sigma_\ell^{-q}\,\Big] \propto 2^{-q\beta\ell/2} \\ \mathrm{P}[\,|Q_\ell/\sigma_\ell| \leq x\,] \propto x \end{array}\right. \implies \begin{array}{c}\text{Adaptive}\\ \text{MLMC}\\ \text{Cost}\end{array} \propto \begin{cases} \varepsilon^{-2} & \beta > \frac{q+1}{q}\gamma \\ \varepsilon^{-2}|\log\varepsilon|^2 & \beta = \frac{q+1}{q}\gamma \\ \varepsilon^{-2(\frac{q+1}{q})(\frac{\gamma}{\beta})} & \beta < \frac{q+1}{q}\gamma. \end{cases}$$



Application: Digital Options

$$\mathrm{P}[\, S_T > K \,], \quad \mathrm{d}S_t = a(S_t)\mathrm{d}t + b(S_t)\mathrm{d}W_t$$



CVA Capital Charge [Giles, H-A, Spence, 2023]

$$\varphi = \mathsf{P}\left[\, \frac{\mathrm{CVA}_H}{B_H} - \mathrm{CVA}_0 > \lambda_\varphi \,\right],$$

$$\mathrm{CVA}_t = B_t\, \mathsf{E}^{\mathsf{Q}}\left[\, \chi_{t \le \tau \le T}\, \mathrm{LGD} \max\left\{ \mathsf{E}^{\mathsf{Q}}\left[\, B_T^{-1} \pi(S_T) \,\middle|\, \mathcal{G}_\tau \,\right], 0 \right\} \,\middle|\, \mathcal{G}_t \,\right].$$

Using a combination of

- Milstein simulation of the assets S_T
- Nested multilevel Monte Carlo estimation
- Unbiased multilevel Monte Carlo sampling
- Variance reduction techniques,

we can express

$$\frac{\mathrm{CVA}_H}{B_H} - \mathrm{CVA}_0 = \mathsf{E}[\, \Delta \mid Z \,],$$

where Z captures all $\mathcal{G}_H$-measurable risk-factors and Δ is a random variable which can be sampled exactly.

CVA Capital Charge [Giles, H-A, Spence, 2023]

$$\varphi = \mathrm{P}[\, U_0(Z) > \lambda_\varphi \,]$$
$$U_0(Z) \coloneqq \mathrm{E}[\, \Delta \mid Z \,]$$

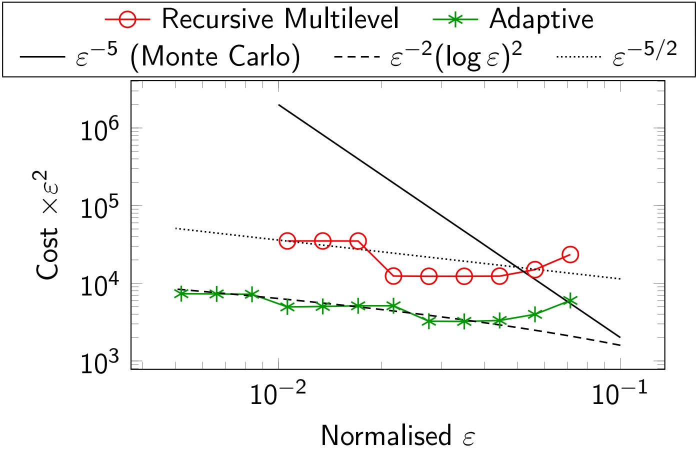
- Approximate

$$U_0(Z) \approx \widehat{U}_{0,\ell}(Z) = \frac{1}{N_\ell} \sum_{n=1}^{N_\ell} \Delta^{(n)}(Z).$$

- Adaptively add more independent samples according to the value of $|\widehat{U}_{0,\ell}(Z)|/\sigma_\ell$, where σ_ℓ^2 is the conditional sample variance.
  - Normalizing by σ_ℓ can mitigate issues caused by unbiased multilevel Monte Carlo leading to fewer moments of $\Delta(Z)$.

CVA: Numerical Results

$$\varphi = \mathrm{P}[\, U_0(Z) > \lambda_\varphi \,]$$



Conclusion

- Multilevel Monte Carlo methods can be extended recursively to systems of repeatedly nested expectations.
- Adaptive sampling provides a general framework to improve multilevel Monte Carlo methods for problems which contain discontinuous functions of approximated random variables.
- A combination of both approaches can provide significant gains over nested Monte Carlo simulation for problems arising in credit risk.

Risk measures:

- The value-at-risk solves $\varphi = \mathrm{P}[\, Q > \lambda_\varphi \,]$. See recent work by [Crepey, Frikha, Louzi, Spence, 2024] which utilizes the adaptive techniques discussed here with a multilevel stochastic approximation techniques to find the quantile λ_φ.
- Conditional value-at-risk is $\mathrm{E}[\, Q \,|\, Q > \lambda_\varphi \,] = \inf_x f(x) = f(\lambda_\varphi)$ for a given φ and $f(x) = x + \mathrm{E}[\, \max\{Q - x, 0\} \,]/\varphi$ [Rockafellar, Uryasev, 1999].

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