Efficient Risk Estimation for the Credit Valuation Adjustment

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Motivation: CVA Capital Charge

$$\text{Adjusted Value}_t = \text{Risk-Neutral Value}_t - \mathrm{CVA}_t$$

- The value of a financial portfolio at time t is reduced according to the credit valuation adjustment CVA_t, to account for the possibility of counterparty default.
- Through the credit valuation adjustment, a financial institution can observe losses due to the event that a counterparty becomes more likely to default - responsible for two thirds of losses from counterparty risk factors in 2008 financial crisis.
- Capital charge based on a value-at-risk formula introduced in Basel III accords to mitigate such risks.

Motivation: CVA Capital Charge (Nested Simulation)

- Given market at time 0 - simulate $\mathcal{G}_H$-measurable market and credit risk factors under the physical measure P at short risk horizon $0 < H \ll 1$.
  - Given risk factors at time H, simulate instances of default τ under the risk-neutral measure Q occurring before contract maturity $T > 0$.
    - Given the (risk-neutral) market state at default time $\tau < T$, simulate random losses $\pi(S_T)$ based on the asset values S_T under the risk-neutral measure.

Value-at-Risk formula:

$$\varphi = \mathsf{P}\left[\frac{\mathrm{CVA}_H}{B_H} - \mathrm{CVA}_0 > \lambda_\varphi \right],$$

$$\mathrm{CVA}_t = B_t\, \mathsf{E}^{\mathsf{Q}}\left[\chi_{t \le \tau \le T}\, \mathrm{LGD} \max\left\{ \mathsf{E}^{\mathsf{Q}}\left[B_T^{-1} \pi(S_T) \,\middle|\, \mathcal{G}_\tau \right], 0 \right\} \,\middle|\, \mathcal{G}_t \right].$$

Overview

Let X, Y, Z be random variables and $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be Lipschitz in both arguments. Consider the system

$$\begin{aligned} \varphi &= \mathsf{P}[\, U_0(Z) > \lambda_\varphi \,] \\ U_0(Z) &= \mathsf{E}[\, f(U_1(Y), Y) \,|\, Z \,] \\ U_1(Y) &= \mathsf{E}[\, X \,|\, Y \,]. \end{aligned}$$

Key features:

- Recursive approximation of nested expectations $U_0(Z)$ and $U_1(Y)$, paired with approximation of the variables X, Y and Z.
- Approximation of discontinuous observables:

$$\varphi = \mathsf{P}[\, Q > 0 \,] = \mathsf{E}[\, \chi_{Q>0} \,].$$

Nested Monte Carlo simulation has $\mathcal{O}(\varepsilon^{-5})$ cost to attain an accuracy ε.

Multilevel Monte Carlo

Want to approximate

$$\mathrm{E}[\,Q\,]$$

given approximate samples $Q \approx Q_\ell$, with

$$\begin{aligned} \mathrm{Cost}(Q_\ell) &\propto 2^{\gamma\ell} \\ |\mathrm{E}[\,Q - Q_\ell\,]| &\propto 2^{-\alpha\ell} \\ \mathrm{Var}[\,Q - Q_\ell\,] &\propto 2^{-\beta\ell}. \end{aligned}$$

Then, let

$$\mathrm{E}[\,\Delta_\ell Q\,] = \begin{cases} \mathrm{E}[\,Q_\ell - Q_{\ell-1}\,] & \ell > 0 \\ \mathrm{E}[\,Q_0\,] & \ell = 0 \end{cases}$$

$$\mathrm{E}[\,Q\,] \approx \mathrm{E}[\,Q_L\,] = \sum_{\ell=0}^{L} \mathrm{E}[\,\Delta_\ell Q\,].$$

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The cost of attaining root mean square error ε is of order

$$\varepsilon^{-2} \begin{cases} 1 & \beta > \gamma \\ |\log \varepsilon|^2 & \beta = \gamma \\ \varepsilon^{-(\gamma-\beta)/\alpha} & \beta < \gamma. \end{cases}$$

Unbiased Multilevel Monte Carlo [Rhee, Glynn, 2015]

$$\mathrm{E}[\,Q\,] = \sum_{\ell=0}^{\infty} \mathrm{E}[\,\Delta_\ell Q\,] = \mathrm{E}\Big[\,(\Delta_\kappa Q)\,2^{\zeta\kappa}/\,C_\zeta\,\Big]$$

where κ is a random, non-negative integer with probability mass

$$\mathrm{P}[\,\kappa = \ell\,] = C_\zeta 2^{-\zeta\ell}.$$

Provided,

$$\mathrm{Cost}(Q_\ell) \propto 2^{\gamma\ell}$$

$$\mathrm{E}[\,|Q - Q_\ell|^q\,] \propto 2^{-q\beta\ell/2},$$

$(\Delta_\kappa Q)\,2^{\zeta\kappa}/\,C_\zeta$ has finite expected sampling cost and p^{th} moment when

$$\gamma < \zeta < \frac{p}{p-1}\frac{\beta}{2} \implies p < \min\left\{q, \frac{1}{1-\beta/2\zeta}\right\}.$$

Example 1: $\beta = 2\gamma = 2, \zeta = (\beta+\gamma)/2 = 3/2, q \to \infty \Rightarrow p < 3$.
Example 2: Same, with
$q = 3-\varepsilon, \zeta < (3-\varepsilon)/(2-\varepsilon) \Rightarrow p < 3-\varepsilon$.

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Let X, Y, Z be random variables and $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ sending $(u, y) \mapsto f(u, y)$ be Lipschitz in u and y. Consider the system

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Key features:

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$$\varphi = \mathrm{P}[\, Q > 0 \,] = \mathrm{E}[\, \chi_{Q>0} \,].$$

Nested Simulation

Consider now the nested pair of expectations

$$U_0 := \mathrm{E}[\, f(U_1(Y),\, Y)\,]$$
$$U_1(Y) := \mathrm{E}[\, X \mid Y\,]$$

Given exact samples of X and Y: Multilevel Monte Carlo with antithetic nested Monte Carlo averages for $\mathrm{E}[\, X \mid Y\,]$.
[Bourgey, De Marco, Gobet, 2020], [Bujok, Hambly, Reisinger, 2015], [Giles, H-A, 2019].

Problem: Exact samples of X and Y are not available. Instead, given $Y = y$ we approximate $X \approx X_k(y)$ using a Milstein scheme. Similarly, $Y \approx Y_\ell$.

Solution: Combine nested 'inner' multilevel Monte Carlo estimate of $U_1(y)$, given $Y = y$, within an 'outer' multilevel Monte Carlo estimate of U_0.

Antithetic Multilevel Difference

Consider now the nested pair of expectations

$$
\begin{aligned}
U_0 &:= \mathrm{E}[\, f(U_1(Y),\ Y)\,] \\
U_1(Y) &:= \mathrm{E}[\, X \mid Y\,] \\
\widehat{U}_{1,\ell}(y) &:= \sum_{k=0}^{\ell} \frac{1}{N_{\ell,k}} \sum_{n=1}^{N_{\ell,k}} \Delta_k^{(n)} X(y) \\
N_{\ell,k} &\propto 2^{\ell-k}.
\end{aligned}
$$

Antithetic multilevel difference:

$$
\Delta_\ell f := f\Big(\widehat{U}_{1,\ell}(Y_\ell), Y_\ell\Big) - \frac{1}{2}\sum_{i=0}^{1} f\Big(\widehat{U}^{(i)}_{1,\ell-1}(Y_{\ell-1}), Y_{\ell-1}\Big),
$$

where

$$
\widehat{U}_{1,\ell}(y) - \frac{1}{2}\sum_{i=0}^{1} \widehat{U}^{(i)}_{1,\ell-1}(y) = \mathcal{O}(\Delta_\ell X).
$$

Convergence

Theorem ([H-A, Spence, 2023])

Assume f is piecewise-twice differentiable and with bounded first and second derivative, and that for $\beta \geq 1$ and $q \geq 2$

$$\begin{aligned} Cost(X_k(\cdot)) + Cost(Y_\ell) &\propto 2^k + 2^\ell \\ \mathrm{E}[\,\|X_k(Y_\ell) - X_{k-1}(Y_\ell)\|^q\,] + \mathrm{E}[\,\|Y_\ell - Y_{\ell-1}\|^q\,] &\propto 2^{-q\beta k/2} + 2^{-q\beta\ell/2} \\ \mathrm{E}[\,\|X_k(Y) - X_k(Y_\ell)\|^q\,] &\propto 2^{-q\beta\ell/2}. \end{aligned}$$

Then,

$$\begin{aligned} \mathrm{Cost}(\Delta_\ell f) &\propto \ell 2^\ell \\ \mathrm{Var}[\,\Delta_\ell f\,] &\propto 2^{-\min\{\beta, 3q/2(q+1)\}\ell}. \end{aligned}$$

Consequently, the cost of estimating U_0 to accuracy ε is of order

$$\varepsilon^{-2}\begin{cases} 1 & \beta > 1 \\ |\log \varepsilon|^3 & \beta = 1. \end{cases}$$

Extensions

- Can be applied recursively to consider repeatedly nested expectations of the form

$$\begin{aligned} U_j(Y_j) &= \mathrm{E}[\, f_{j+1}(U_{j+1}(Y_{j+1}), Y_{j+1}) \mid Y_j \,] \\ U_{T-1}(Y_{T-1}) &= \mathrm{E}[\, f_T(Y_T) \mid Y_{T-1} \,]. \end{aligned}$$

  - Bermudan option pricing/optimal control.
- Can randomise the approximation level ℓ in the terms $\Delta_\ell X$ and $\Delta_\ell f$ to obtain unbiased estimates of $U_1(Y)$ and U_0.
  - [Zhou, Wang, Blanchet, Glynn, 2022], [Syed, Wang, 2023].
  - Reduces the number of finite moments - leading to large variance and sampling cost for repeatedly nested expectations as above.
- Can be extended to include antithetic path simulation of Y.

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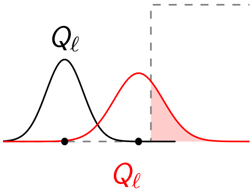
Framework

$$\mathrm{P}[\,Q > 0\,] = \mathrm{E}[\,\chi_{Q>0}\,] \approx \mathrm{E}[\,\chi_{Q_0>0}\,] + \sum_{\ell=1}^{L} \mathrm{E}\left[\,\chi_{Q_\ell>0} - \chi_{Q_{\ell-1}>0}\,\right]$$

Theorem

For root mean square error ε and (random), positive-valued, normalising factor σ_ℓ

$$\begin{array}{r} \mathit{Cost}(Q_\ell) \propto 2^{\gamma\ell} \\ \mathrm{E}\left[\,|Q - Q_\ell|^q \sigma_\ell^{-q}\,\right] \propto 2^{-q\beta\ell/2} \\ \mathrm{P}[\,|Q_\ell/\sigma_\ell| \leq x\,] \propto x \end{array} \implies \begin{array}{c}\text{MLMC}\\\text{Cost}\end{array} \propto \begin{cases} \varepsilon^{-2} & \beta > \frac{q+1}{q} 2\gamma \\ \varepsilon^{-2} |\log \varepsilon|^2 & \beta = \frac{q+1}{q} 2\gamma \\ \varepsilon^{-1-2(\frac{q+1}{q})(\frac{\gamma}{\beta})} & \beta < \frac{q+1}{q} 2\gamma. \end{cases}$$



Previous Research

- **Explicit smoothing $\chi_{x>0} \approx g(x)$:**

  [Giles, Nagapetyan, Ritter, 2015].

- **Numerical smoothing:**

  [Bayer, Hammouda, Tempone, 2023], [Giles, Debrabant, Rößler, 2019].

- **Path branching:**

  [Giles, H-A, 2022].

- **Quasi-Monte Carlo:**

  [Xu, He, Wang, 2020].

- **Adaptivity:**
  - For partial differential equations with random coefficients [Elfverson, Hellman, Målqvist, 2016].
  - For nested expectations $Q = \mathrm{E}[\, X \mid Y \,]$ [Broadie, Du, Moallemi, 2011], [Giles, H-A, 2019].

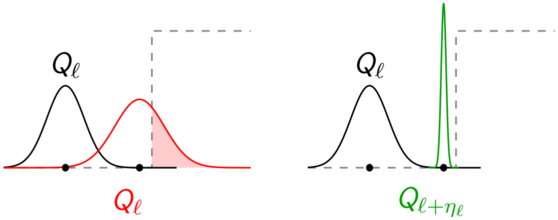
Adaptivity

$$\mathsf{E}[\,\chi_{Q>0}\,] \approx \mathsf{E}[\,\chi_{Q_0>0}\,] + \sum_{\ell=1}^{L} \mathsf{E}\Big[\,\chi_{Q_{\ell+\eta_\ell}>0} - \chi_{Q_{\ell-1+\eta_{\ell-1}}>0}\,\Big]$$

Theorem ([H-A, Spence, Teckentrup, 2022])

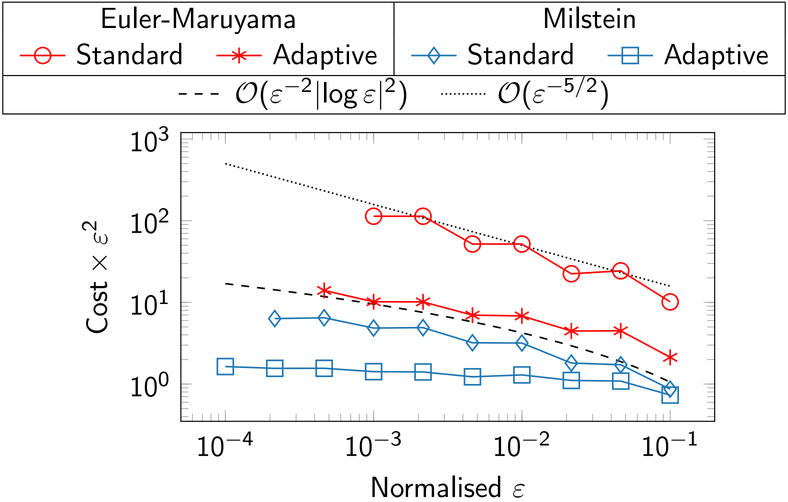
Let η_ℓ *be such that* $|Q_{\ell+\eta_\ell}| \geq \sigma_{\ell+\eta_\ell} 2^{\gamma(\ell(1-r)-\eta_\ell)/r}$ *or* $\eta_\ell = \ell$. *Then, for a root-mean-square error* $\varepsilon > 0$

$$\begin{array}{c} \mathit{Cost}(Q_\ell) \propto 2^{\gamma\ell} \\ \mathsf{E}\Big[\,|Q - Q_\ell|^q \sigma_\ell^{-q}\,\Big] \propto 2^{-q\beta\ell/2} \\ \mathsf{P}[\,|Q_\ell/\sigma_\ell| \leq x\,] \propto x \end{array} \implies \begin{array}{c}\text{Adaptive}\\ \text{MLMC}\\ \text{Cost}\end{array} \propto \begin{cases} \varepsilon^{-2} & \beta > \frac{q+1}{q}\gamma \\ \varepsilon^{-2}|\log \varepsilon|^2 & \beta = \frac{q+1}{q}\gamma \\ \varepsilon^{-2\left(\frac{q+1}{q}\right)\left(\frac{\gamma}{\beta}\right)} & \beta < \frac{q+1}{q}\gamma. \end{cases}$$



Application: Digital Options

$$\mathrm{P}[\, S_T > K \,], \quad \mathrm{d}S_t = a(S_t)\mathrm{d}t + b(S_t)\mathrm{d}W_t$$



CVA Capital Charge [Giles, H-A, Spence, 2023]

$$\varphi = \mathsf{P}\Big[\, \frac{\mathrm{CVA}_H}{B_H} - \mathrm{CVA}_0 > \lambda_\varphi \,\Big],$$
$$\mathrm{CVA}_t = B_t\, \mathsf{E}^{\mathsf{Q}}\Big[\, \chi_{t\le\tau\le T}\, \mathrm{LGD} \max\Big\{ \mathsf{E}^{\mathsf{Q}}\big[\, B_T^{-1}\pi(S_T) \,\big|\, \mathcal{G}_\tau \,\big], 0 \Big\} \,\Big|\, \mathcal{G}_t \,\Big].$$

Using a combination of

- Milstein simulation of the assets S_T
- Nested multilevel Monte Carlo estimation
- Unbiased multilevel Monte Carlo sampling
- Variance reduction techniques,

we can express

$$\frac{\mathrm{CVA}_H}{B_H} - \mathrm{CVA}_0 = \mathsf{E}[\, \Delta \,|\, Z \,],$$

where Z captures all $\mathcal{G}_H$-measurable risk-factors and Δ is a random variable which can be sampled exactly.

CVA Capital Charge [Giles, H-A, Spence, 2023]

$$\varphi = \mathrm{P}[\, U_0(Z) > \lambda_\varphi \,]$$
$$U_0(Z) \coloneqq \mathrm{E}[\, \Delta \mid Z \,]$$

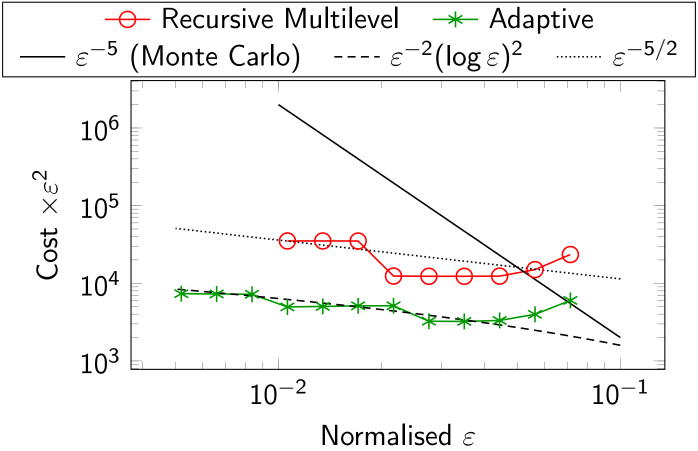
- Approximate

$$U_0(Z) \approx \widehat{U}_{0,\ell}(Z) = \frac{1}{N_\ell} \sum_{n=1}^{N_\ell} \Delta^{(n)}(Z).$$

- Adaptively add more independent samples according to the value of $|\widehat{U}_{0,\ell}(Z)|/\sigma_\ell$, where σ_ℓ^2 is the conditional sample variance.
  - Normalizing by σ_ℓ can mitigate issues caused by unbiased multilevel Monte Carlo leading to fewer moments of $\Delta(Z)$.

CVA: Numerical Results

$$\varphi = \mathrm{P}[\, U_0(Z) > \lambda_\varphi \,]$$



Conclusion

- Multilevel Monte Carlo methods can be extended recursively to systems of repeatedly nested expectations.
- Adaptive sampling provides a general framework to improve multilevel Monte Carlo methods for problems which contain discontinuous functions of approximated random variables.
- A combination of both approaches can provide significant gains over nested Monte Carlo simulation for problems arising in credit risk.

Risk measures:

- The value-at-risk solves $\varphi = \mathrm{P}[\, Q > \lambda_\varphi \,]$. See recent work by [Crepey, Frikha, Louzi, Spence, 2024] which utilizes the adaptive techniques discussed here with a multilevel stochastic approximation techniques to find the quantile λ_φ.
- Conditional value-at-risk is $\mathrm{E}[\, Q \,|\, Q > \lambda_\varphi \,] = \inf_x f(x) = f(\lambda_\varphi)$ for a given φ and $f(x) = x + \mathrm{E}[\, \max\{Q - x, 0\} \,]/\varphi$ [Rockafellar, Uryasev, 1999].

- C. Bayer, C. B. Hammouda, and R. Tempone, *Multilevel Monte Carlo combined with numerical smoothing for robust and efficient option pricing and density estimation*, arXiv preprint arXiv:2003.05708, (2023).
- ——, *Numerical smoothing with hierarchical adaptive sparse grids and quasi-Monte Carlo methods for efficient option pricing*, Quantitative Finance, 23 (2023), pp. 209–227.
- F. Bourgey, S. De Marco, E. Gobet, and A. Zhou, *Multilevel Monte Carlo methods and lower-upper bounds in initial margin computations*, Monte Carlo Methods and Applications, 26 (2020), pp. 131–161.
- M. Broadie, Y. Du, and C. Moallemi, *Efficient risk estimation via nested sequential simulation*, Management Science, 57 (2011), pp. 1172–1194.

K. Bujok, B. Hambly, and C. Reisinger, *Multilevel simulation of functionals of Bernoulli random variables with application to basket credit derivatives*, Methodology and Computing in Applied Probability, 17 (2015), pp. 579–604.

D. Elfverson, F. Hellman, and A. Målqvist, *A multilevel Monte Carlo method for computing failure probabilities*, SIAM/ASA Journal on Uncertainty Quantification, 4 (2016), pp. 312–330.

M. Giles, K. Debrabant, and A. Rößler, *Numerical analysis of multilevel Monte Carlo path simulation using the Milstein discretisation*, Discrete and Continuous Dynamical Systems, 24 (2019), pp. 3881–3903.

M. Giles and A.-L. Haji-Ali, *Multilevel nested simulation for efficient risk estimation*, SIAM/ASA Journal on Uncertainty Quantification, 7 (2019), pp. 497–525.

——, *Multilevel path branching for digital options*, arXiv pre-print: 2209.03017, (2022).

- M. Giles, A.-L. Haji-Ali, and J. Spence, *Efficient risk estimation for the credit valuation adjustment*, arXiv preprint arXiv:2301.05886, (2023).
- M. Giles, T. Nagapetyan, and K. Ritter, *Multilevel Monte Carlo approximation of distribution functions and densities*, SIAM/ASA Journal on Uncertainty Quantification, 3 (2015), pp. 267–295.
- A.-L. Haji-Ali, J. Spence, and A. Teckentrup, *Adaptive multilevel Monte Carlo for probabilities*, SIAM Journal on Numerical Analysis, 60 (2022), pp. 2125–2149.
- F. Jonsson, *On the heavy-tailedness of Student's t-statistic*, Bernoulli, 17 (2011), pp. 276–289.
- C.-H. Rhee and P. Glynn, *Unbiased estimation with square root convergence for SDE models*, Operations Research, 63 (2015), pp. 1026–1043.

- Y. Syed and G. Wang, *Optimal randomized multilevel Monte Carlo for repeatedly nested expectations*, Proceedings of the 40th International Conference on Machine Learning, PLMR, 202 (2023), pp. 33343–33364.
- Z. Xu, Z. He, and X. Wang, *Efficient risk estimation via nested multilevel quasi-Monte Carlo simulation*, Journal of Computational and Applied Mathematics, 443 (2024), p. 115745.
- Z. Zhou, G. Wang, J. Blanchet, and P. Glynn, *Unbiased optimal stopping via the MUSE*, Stochastic Processes and their Applications, 166 (2023), p. 104088.